Fully Homomorphic Encryption

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Outline

- Motivation
- A naïve scheme and its problems
- Existing Scheme and its implementation
- Problems
- Future Work

The Goal of FHE

- I want to delegate processing of my data, without giving away access to it.

  -Craig Gentry(2009)

Application 1 – Cloud Computing

- Data stored on cloud in encrypted form
- You want to perform SECRET operations on the data
- Encrypt simple queries to _queries_
- Send _queries_ to cloud
- Cloud performs _queries_ on encrypted data and sends back encrypted results
- Decrypt them to get actual results

Application 2 – Private Google Search

- You don't want Google to know your SECRET queries

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- You don't want Google to know your SECRET queries
- Submit encrypted queries
- Get encrypted results
- Decrypt results

Our Goal(s)

- Perform operations of data without knowing the contents EFFICIENTLY
- Performing attacks on the existing scheme,
  especially SIDE-CHANNEL ATTACKS.

A simple scheme

- Shared secret key: odd number p
- To *encrypt* a bit m in {0,1}:
- Choose at random small r, large q
- Output c = m + 2r + pq

noise

m = LSB of distance to nearest multiple of p

- To *decrypt* c:
- Output m = (c mod p) mod 2

A simple scheme

- Shared secret key: odd number p(=101)
- To *encrypt* a bit m in {0,1}:
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A simple scheme

- Shared secret key: odd number p(=101)
- To *encrypt* a bit m in {0,1}: (say m=1)
- Choose at random small r(=5), large q(=9)
- Output c = m + 2r + pq

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A simple scheme

- Shared secret key: odd number p(=101)
- To *encrypt* a bit m in {0,1}: (say m=1)
- Choose at random small r(=5), large q(=9)
- Output c = m + 2r + pq= 1 + 10 + 909= 920

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  = (920 mod 101) mod 2 = 11 mod 2 = 1

Homomorphism?

- $c_1 = m_1 + 2r_1 + pq_1$ $c_2 = m_2 + 2r_2 + pq_2$

Noise

- $c_1 + c_2 = (m_1 + m_2) + 2(r_1 + r_2) + p(q_1 + q_2)$
- $(c_1 + c_2 \bmod p) \bmod 2 = m_1 + m_2 \bmod 2$ **= m1 XOR m2**

Noise

- $c_1 . c_2 = (m_1 + 2r_1).(m_2 + 2r_2) + p(q')$
- $(c_1 . c_2 \bmod p) \bmod 2 = m_1 . m_2 \bmod 2$ **= m1 AND m2**

Homomorphism?

- $c_1=m_1+2r_1+pq_1$ $c_2=m_2+2r_2+pq_2$
- $c_1=1+2.5+9.101$ $c_2= 1+2.7+8.101$
- $=11+909$ $= 15+808$
- $c_1.c_2=11.15 + 295.101=165 + 295.101$

- $c_1.c_2$ mod 101 = 165 mod 101 = 64
- ($c_1.c_2$ mod 101) mod 2 = 0 (Incorrect!)

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Noise Problem

- Problem arises when noise becomes comparable to p
- When this happens, cipher-texts could be decrypted, and again encrypted with fresh noise, which is always small

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- When this happens, cipher-texts could be decrypted, and again encrypted with fresh noise, which is always small

- Wouldn’t that compromise privacy?

Need: A Bootstrappable Scheme

- A scheme which can handle its own decryption function
- If such a scheme can be designed, cipher texts encrypted under one key, can be encrypted for another level with another key, and then one level of encryption removed

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- We will come back to this!

Gentry's FHE scheme

- KeyGen(λ)
- Encrypt(pk, m)
- Decrypt(sk, m)
- Evaluate(pk, f, c_1, … , c_t)
- Recrypt(pk_2, D_{ε}, sk_1, c_1)

Parameter Declaration

- Read security parameter λ
- Set $N \leftarrow \lambda$, $P \leftarrow \lambda^2$, $Q \leftarrow \lambda^5$
- Randomly select two integer parameters $0 < \alpha < \beta$

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KeyGen(λ)

- Generates pk, sk

- p is a random P-bit odd integer
- Generate a set $\mathbf{y}=\{y_1, \ldots y_\beta\}: y_i \in [0, 2)$
- For a sparse subset S of size α,
  $\sum y_S = (1/p) \bmod 2$

- sk ← s, where s $=\{0,1\}^\beta$ is an encoding of S
- pk ← (p, $\mathbf{y}$)

Implementation Technique

- Structure **publicKey** defined with one integer (p) and an array (**y**) of reals for pk.
- Each element is subset solution is set at (1/p+2(rand() mod α))/α
- Every other element of **y** is set randomly

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Encrypt(pk, m)

- Generate an N-bit integer m' such that m' =m mod 2
- Generate a random Q-bit integer q
- Set c = m' + (pk.p)*q
- Generate a set **z**:$z_i \leftarrow c^* y_i$ mod 2
- Return **c** ← (c, **z**)

Implementation Technique

- Required a *mod2* function, which can compute values of reals modulo 2.
- Necessary for post-processing **y** to compute **z**.

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Decrypt(sk, c)

- To return (c mod p) mod 2
- Equivalent to LSB(c) XOR LSB($\lfloor(c/p)\rceil$)
- $\lfloor.\rceil$ returns nearest integer
- $\Sigma\,(sk_i * z_i) = c\{\Sigma\,(sk_i * y_i)\} = c(1/p) \bmod 2$

Implementation Technique

- Function *nearest_int*
- Function *LSB*

Gentry's FHE scheme

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- Encrypt(pk, m)
- Decrypt(sk, m)
- Evaluate(pk, f, c_1, … , c_t)

- Recrypt(pk_2, D_ε, sk_1, c_1)

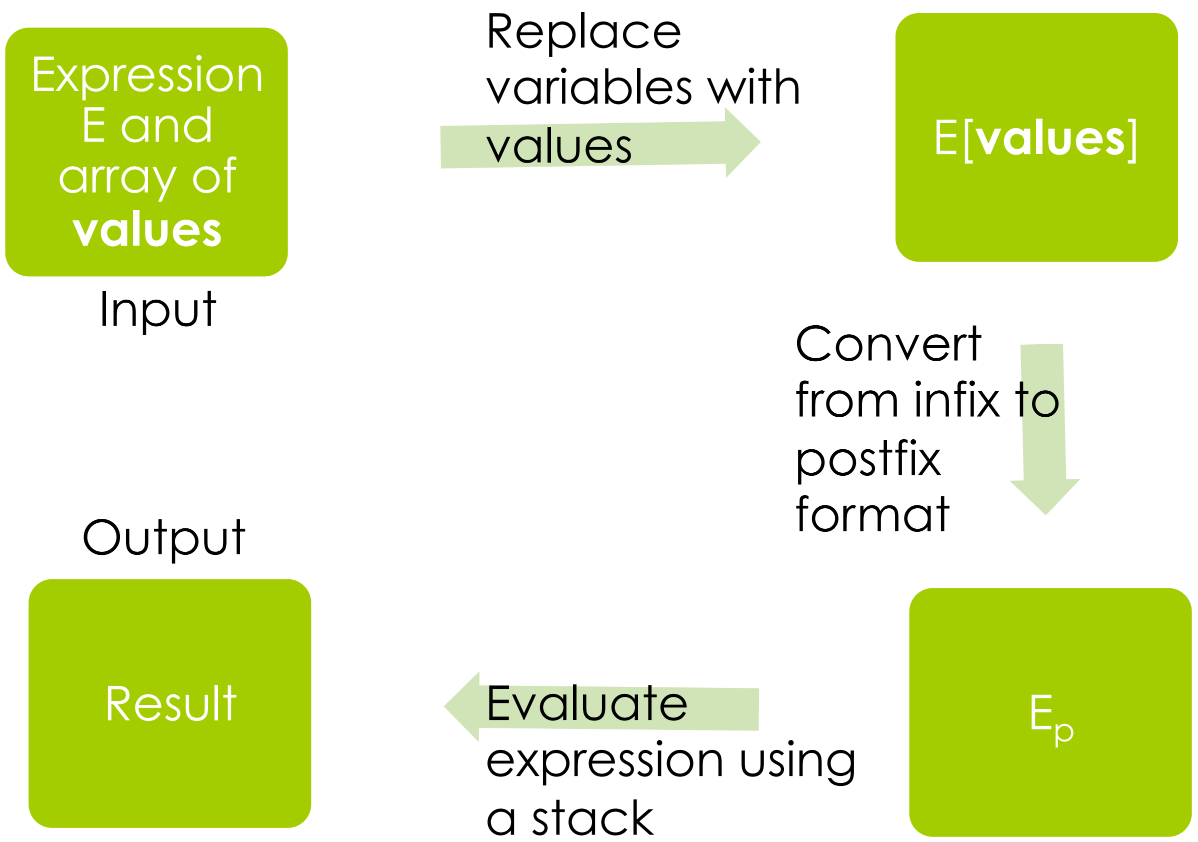
Evaluate(pk, f, c_1, …, c_t)

- Takes in boolean function with only ANDs and XORs
- Replaces AND with multiplication
- Replaces XOR with addition
- Returns $c \leftarrow f(c_1, \ldots, c_t)$

Implementation Technique

- Each c_i is of type **publicKey**.
- Technically, computes $c.p \leftarrow f(c_1.p, \ldots, c_t.p)$
- c.**y** is computed as $c.y_i \leftarrow pk.y_i * c.p$
- An *expression evaluator* was developed using stacks

Expression Evaluator



Gentry's FHE scheme

- KeyGen(λ)
- Encrypt(pk, m)
- Decrypt(sk, m)
- Evaluate(pk, f, c_1, … , c_t)
- Recrypt(pk_2, D_ε, sk_1, c_1)

Recrypt(pk_2, D, $\mathbf{sk_1}$, c_1)

- D is the boolean expression for the decryption function
- $\mathbf{sk_1}$ is a vector of cipher-texts , where
  $\mathbf{sk_1}[i] \leftarrow \text{Encrypt}(pk_2, sk_1[i])$
- c_1 is a cipher-text encrypted under pk_1
- Compute $\mathbf{c_1}$: $\mathbf{c_1}[i] \leftarrow \text{Encrypt}(pk_2, <c_1>_i)$
- Return $c \leftarrow \text{Evaluate}(pk_2, D, \mathbf{sk_1}, \mathbf{c_1})$

Implementation Issues

- Formulation of D using naïve integer methods
- Published implementations till date [Gentry' 11], [Smart' 09] have used lattice based methods

Timing Measurements

Dimension	KeyGen	Encrypt	Decrypt
2^3	0.405 ms	0.145 ms	0.125 ms
2^5	0.421 ms	0.337 ms	3.43 ms
2^7	0.422 ms	4.2 ms	16.36 ms
2^9	0.438 ms	33.37 ms	24.54 ms
2^{11}	0.437 ms	187.02 ms	89.16 ms
2^{13}	0.434 ms	474.29 ms	215.94 ms
2^{15}	0.433 ms	0.99 sec	0.5 sec

Short Term Goals

- Generalization of input by writing a convertor for boolean functions to AND-XOR form
- Use lattice-based methods to implement Recrypt
- Extensive testing

Long Term Goals

- Improve time and memory complexity of scheme. Current implementations are not practical
- Explore the possibilities of side-channel attacks on this scheme